1. Let $f:R\to R$ be a differentiable function. Prove that there is a point $t\in(0,1)$ such that

$$f(1)-f(0)=\frac{\pi}{2}\sqrt{1-t^2}\,f'(t).$$

(Hint: Consider $g(x)=f(\sin x)$ for all $0\le x\le\frac{\pi}{2}$)

<u>Solution</u>

For the function $g(x)=f(\sin x)$, $0\le x\le\frac{\pi}{2}$, apply the Mean Value Theorem.
$g\left(\frac{\pi}{2}\right)-g(0)=g'(c)\left(\frac{\pi}{2}-0\right)$
$g'(x)=f'(\sin x)\cos x$
Then, $f(1)-f(0)=\frac{\pi}{2}f'(\sin c)\cdot\cos c$.
Let $t=\sin c$.
$\cos c=\sqrt{1-\sin^2 c}=\sqrt{1-t^2}$
$f(1)-f(0)=\frac{\pi}{2}\sqrt{1-t^2}\,f'(t)$